RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

B.A./B.Sc. FIFTH SEMESTER EXAMINATION, DECEMBER 2014

THIRD YEAR

MATHEMATICS (Honours)

Date : 23/12/2014

Time : 11 am – 3 pm

Paper : VI

Full Marks : 100

[Use a separate Answer book for each Group]

Group – A

Unit – I

(*Answer **any six** questions*) [6×5]

1. Let $f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

   Examine for continuity at (0,0), existence of directional derivative at (0,0) and differentiability at (0,0). [2+2+1]

2. Suppose E is an open set in R^n, f maps E into R^m, f is differentiable at $x_0\in E$, g maps an open set containing f(E) into R^k, and g is differentiable at $f(x_0)$. Then prove that F of E into R^K defined by $F(x)=g(f(x))$ is differentiable at x_0 and $F'(x_0)=g'(f(x_0))f'(x_0)$. [5]

3. a) Let f and g be functions defined on $\mathbb{R}$, f(0) = 0, $f'(0)\neq 0$, with continuous derivatives. Consider the equation f(x) = tg(x), $x\in\mathbb{R}$. Show that in a suitable interval $|t|<\delta$, there is a unique continuous function x = x(t), that solves the above equation and satisfies x(0) = 0.

   b) Let f(x,t) have continuous first order partial derivatives such that $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial t}$. Let $f(x,0)>0$ for all x. Show that $f(x,t)>0$ for all x & t. [3+2]

4. Let $f:S\to\mathbb{R}$ where $S(\subset\mathbb{R}^2)$ be an open set and $(a,b)\in S$. Let $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ be differentiable at (a,b). Prove that $f_{xy}(a,b)=f_{yx}(a,b)$. [5]

5. State and prove Taylor's theorem for a function of two variables, $f:\mathbb{R}^2\to\mathbb{R}$. [5]

6. a) Let functions u = f(x,y,z), v = g(x, y, z) have continuous first order partial derivatives in a region R and there exists a functional relation F(u,v) = 0.

   Prove that all 2-rowed determinants of the Jacobian matrix $\begin{pmatrix}u_x & u_y & u_z\\ v_x & v_y & v_z\end{pmatrix}$ must vanish.

   b) Given $f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)f(y)}$ where x, y are independent variables, f(t) is a differentiable function and f(0) = 0 and $f(x)f(y)\neq 1$, show that $f(t)=\tan\alpha t$, where α is a constant. [2+3]

7. Find the point on the parabola $y^2 = 2x$, z = 0 which is nearest the plane z = x + 2y +8. Show that the minimum distance is $\sqrt{6}$. [4+1]

8. If x = f(r, s), y = g(r, s), $J=\dfrac{\partial(f,g)}{\partial(r,s)}\neq 0$ and if h is a function of r, s; show that

$$\frac{\partial h}{\partial x}=\frac{\partial(h,g)}{\partial(r,s)}.\frac{1}{J},\quad \frac{\partial h}{\partial y}=\frac{\partial(f,h)}{\partial(r,s)}.\frac{1}{J}.$$ [5]

9. Let f and g are homogeneous functions of degree n, having continuous first order partial derivatives. If $f=\lambda u$, $g=\lambda v$ where λ is homogeneous function of degree $m(\neq n)$, having continuous first order partial derivatives, prove that

$$\frac{\partial(f,g)}{\partial(x,y)}=\frac{n\lambda^2}{n-m}\cdot\frac{\partial(u,v)}{\partial(x,y)}$$ [5]

Unit – II

(*Answer **any four** questions*) [4×5]

10. a) Suppose a particle moves along a curve $\vec{r}=(t^3-4t)\hat{i}+(t^2+4t)\hat{j}+(8t^2-3t^3)\hat{k}$.
Find the magnitudes of the tangential and normal components of its acceleration when t = 2. [4]

b) If $\frac{d\vec{a}}{dt}=\lambda\vec{b}$ and $\frac{d\vec{b}}{dt}=\mu\vec{a}$, then $\frac{d}{dt}(\vec{a}\times\vec{b})$ is

i) $\lambda\mu(\vec{a}\times\vec{b})$ ii) $\lambda\mu(\vec{b}\times\vec{a})$ iii) $\frac{\lambda}{\mu}(\vec{a}\times\vec{b})$ iv) $\vec{0}$

Justify your answer. [1]

11. a) If $\vec{f}$ is a differentiable vector function then prove that curl curl $\vec{f}$ = grad div $\vec{f}-\vec{\nabla}^2\vec{f}$. [4]

b) If $\vec{f}=x^2y\,\hat{i}+y^2z\,\hat{j}+3z^2x\,\hat{k}$ for all (x,y,z) $\in R^3$ and $\vec{\nabla}=\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$ then the value of $\vec{\nabla}.(\vec{\nabla}\times\vec{f})$ at (1,1,1) is

i) 3 ii) 6 iii) 9 iv) 0 [1]

12. Suppose $\vec{A}=(2xy+1+3x^2z^2)\hat{i}+x^2\hat{j}+2x^3z\hat{k}$. If C is a smooth curve in R^3 from (–1,0,1) to (1,1,–1) find the value of $\int_C \vec{A}.d\vec{r}$, where $d\vec{r}=dx\hat{i}+dy\hat{j}+dz\hat{k}$.

Show that there is a differentiable function ϕ such that $\vec{A}=\vec{\nabla}\varphi$ and find it. [3+2]

13. Evaluate $\int_S \vec{n}\,.\,\text{curl}\,\vec{F}\;dS$ where $\vec{F}=y\vec{i}+z\vec{j}+x\vec{k}$ and S is the portion of the surface

$x^2+y^2-2ax+az=0$ above the plane z = 0 and verify Stoke's theorem. [3+2]

14. Verify Green's theorem in a plane for $\oint_C\{(y-\sin x)dx+\cos x\,dy\}$ where C represents the triangle with vertices $(0,0),(\pi/2,0)$ and $(\pi/2,2)$. [5]

15. a) Suppose V is the volume bounded by a smooth closed surface and φ is a scalar point function with continuous derivatives. Prove that $\iiint_V \vec{\nabla}\varphi dv=\iint_S \varphi\vec{n}dS$ where $\vec{n}$ is the positive (outward drawn) normal to S. [4]

b) For any closed surface S, the value of $\iint_S \vec{n}\,dS$ is

i) $\hat{i}$ ii) $\hat{j}$ iii) $\hat{k}$ iv) $\vec{O}$

Justify your answer. [1]

Group – B

(*Answer **any two** questions from **Q.No. 16-18***) [2×15]

16. a) Find whether a given straight line is, at any point of its length, a principal axis of a material system. If the line is a principal axis, find the other two principal axes at that point. [5+2]

b) A rod revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α. Show that ω is given by $\omega^2 = \frac{3g}{4a\cos\alpha}$, where 2a is the length of the rod.

Prove also that the direction of the reaction at the hinge makes with the vertical an angle $\tan^{-1}\left(\frac{3}{4}\tan\alpha\right)$. [6+2]

17. a) A solid homogeneous cone of height h and semi-vertical angle α oscillates about a diameter of its base. Show that the length of the simple equivalent pendulum is $\frac{1}{5}h(2+3\tan^2\alpha)$. [7]

b) In two dimensional motion of a rigid body, prove with usual notations that its kinetic energy is $\frac{M}{2}\left[\dot{\bar{x}}^2 + \dot{\bar{y}}^2 + K^2\dot{\theta}^2\right]$. [8]

18. a) Three equal rods AB, BC, CD are freely jointed and placed in a straight line on a smooth table. The rod AB is struck at its end A by a blow which is perpendicular to its length. Show that the velocity of the centre of AB is 19 times that of CD. [5]

b) Obtain equations of motion of a rigid body under impulsive forces in two dimensions. [5]

c) An elliptic area, of eccentricity e, is rotating with angular velocity ω about one latus rectum. Suddenly this latus rectum is loosed and the other is fixed. Show that the new angular velocity is $\frac{1-4e^2}{1+4e^2}\omega$. [5]

(*Answer* ***__any two__*** *questions from* ***__Q.No. 19-21__***) [2×7]

19. Show that if T be the time of describing an arc bounded by a focal chord of a parabolic orbit under Newtonian law, then T varies as $(\text{focal chord})^{3/2}$. [7]

20. A heavy particle slides down a rough cycloid of which the coefficient of fraction is μ. Its base is horizontal and vertex downwards. Show that if it starts from rest at the cusp and comes to rest at the vertex, then $\mu^2 e^{\mu\pi} = 1$. [7]

21. The volume of a spherical raindrop falling freely increases at each instant by an amount equal to μ times its surface area at that instant. If the initial radius of the raindrop be 'a', then show that its radius is doubled when it has fallen through a distance $\frac{9a^2g}{32\mu^2}$. [7]

(*Answer* ***__either Q.No. 22 or 23__***) [1×6]

22. A particle describes an ellipse of eccentricity e about a centre of force at a focus. When the particle is at one end of a minor axis, its velocity is doubled. Prove that the new path is a hyperbola of eccentricity $\sqrt{9-8e^2}$. [6]

23. If the planet were suddenly stopped in its orbit, supposed circular, then show that it would fall into the sun in a time which is $\frac{\sqrt{2}}{8}$ times the period of the planet's revolution. [6]

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